§1.1 集合及其运算

常见集合：$\mathbb{N}, \mathbb{N}^+, \mathbb{Z}, \mathbb{Q}, \mathbb{R}$

$$C[a,b] = \{ f(x) : [a,b] \to \mathbb{R}, f \in C[a,b] \}$$

$$E[f>a] = \{ x : x \in E, f(x) > a \}$$

集族：$\{A_\lambda : \lambda \in \Lambda\}$　集列：$\{A_n : n \in \mathbb{N}\}$

单调增/减列：$A_1 \supset A_2 \supset \cdots \supset A_n \supset \cdots$

$A_1 \subset A_2 \subset \cdots \subset A_n \subset \cdots$

例1　$A_k = [-1+\frac{1}{k}, 1-\frac{1}{k}]$, $k=1,2,\cdots$，求 $\bigcup_{k=1}^{\infty} A_k$, $\bigcap_{k=1}^{\infty} A_k$

Thm.　$S \setminus (\bigcup_{\lambda\in\Lambda} A_\lambda) = \bigcap_{\lambda\in\Lambda} (S \setminus A_\lambda)$

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例2　$A_k = \begin{cases} [0, 2-\frac{1}{k-\frac{1}{3}}), & k\text{奇} \\ [0, 1+\frac{3}{k}), & k\text{偶} \end{cases}$

$\bigcup_{k=1}^{\infty} A_{2k-1} = [0,2)$，$\bigcap_{k=1}^{\infty} A_{2k} = [0,1]$

$\bigcup_{k=1}^{\infty} A_k = [0, \frac{5}{2})$，$\bigcap_{k=1}^{\infty} A_k = [0, \frac{1}{2})$

Def.　$\overline{\lim_{n\to\infty}} A_n = \bigcap_{N=1}^{\infty} \bigcup_{n=N}^{\infty} A_n$，$\underline{\lim_{n\to\infty}} A_n = \bigcup_{N=1}^{\infty} \bigcap_{n=N}^{\infty} A_n$

Thm.

$x = \overline{\lim_{n\to\infty}} A_n \Leftrightarrow \forall N, \exists n > N \text{ s.t. } x \in A_n$

$\Leftrightarrow \exists N_k, N_k \to \infty, \forall N \in N_k, \exists n > N, \text{ s.t. } x \in A_n$

$x = \underline{\lim_{n\to\infty}} A_n \Leftrightarrow \exists N, \forall n > N \text{ s.t. } x \in A_n$

考虑 $B_N = \bigcup_{n=N}^{\infty} A_n$，则 $\overline{\lim_{n\to\infty}} A_n = \bigcap_{N=1}^{\infty} B_N$，其中 B_N 为递减列

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Prop. $\bigcap_{n=1}^{\infty} A_n \subset \underline{\lim}_{n\to\infty} A_n \subset \overline{\lim}_{n\to\infty} A_n \subset \bigcup_{n=1}^{\infty} A_n$

Thm. $\{A_n\}$存在极限 $\Leftrightarrow \underline{\lim}_{n\to\infty} A_n = \overline{\lim}_{n\to\infty} A_n$

例3 (1) 记$A_{2n} = E$, $A_{2n-1} = F$, 则 $\overline{\lim}_{n\to\infty} A_n = E \cup F$, $\underline{\lim}_{n\to\infty} A_n = E \cap F$

(2) $A_{2n} = (-1, 1-\frac{1}{n})$, $A_{2n+1} = (0, 2+\frac{1}{n}]$

$\overline{\lim}_{n\to\infty} A_n = (-1, 2]$, $\underline{\lim}_{n\to\infty} A_n = (0, 1)$

Def. $E[f>a] = \{x \in E \mid f(x) > a\}$

Thm. (1) $E[f>a] = \bigcup_{k=1}^{\infty} E[f > a + \frac{1}{k}]$

(2) $x \in E[f_n \to f] \Leftrightarrow x \in \bigcap_{k=1}^{\infty} \bigcup_{N=1}^{\infty} \bigcap_{n=N}^{\infty} E[|f_n - f| < \frac{1}{k}]$